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LETTER TO THE EDITOR

# The statistical distribution function of the $q$-deformed harmonic oscillator

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**Abstract.** By using the double-time Green function technique, we derive explicitly the exact statistical distribution function of the $q$-deformed harmonic oscillator, and find that it is highly different from the usual Bose and Fermi functions for $q \neq 1$.

Recently there is a great deal of interest in the study of the quantum group $SU_q(2)$ in the field of mathematical physics. It is known presently that a new boson realization of the quantum group can be achieved by means of a $q$-deformation of the quantum harmonic oscillator [1–3]. Through defining the '$q$-creation' operator $a_q^+$ and the '$q$-annihilation' operator $a_q$, the commutation relations for the quantum group can be presented by the two operators. With the help of the $q$-commutator for $SU_q(2)$, in [1–3], the $q$-momentum and $q$-position operators of the $q$-oscillator were well defined, and the Hamiltonian of the system was presented, and also the eigenvalues of the $q$-Hamiltonian was already given, and so on. In turn a question arises naturally: What is the explicit form of the $q$-deformation of the statistical distribution function $f_q$ for the $q$-oscillator? In this letter, we will calculate such an $f_q$ by means of the double-time Green function technique.

Usually, the double-time Green functions (retarded or advanced) describe the linear response of a physical quantity to the applied fields in a certain physical system, which are closely related to correlation functions which are responsible for the fluctuations of the system. The retarded ($\rho = +1$) and the advanced ($\rho = -1$) Green functions are generally defined by [4]

$$G_\rho(t-t') = -\frac{\mathrm{i}}{2\hbar}\{(\rho+1)\theta(t-t')+(\rho-1)\theta(t'-t)\}\langle[A(t), B(t')]_\eta\rangle \qquad (1)$$

where $A(t) = \exp(\mathrm{i}Ht)A\exp(-\mathrm{i}Ht)$, $\langle\hat{\theta}\rangle = Z^{-1}\,\mathrm{Tr}(\mathrm{e}^{-\beta H}\hat{\theta})$, $Z = \mathrm{Tr}(\mathrm{e}^{-\beta H})$, and $[A, B]_\eta = AB + \eta BA$; $H$ is the Hamiltonian of the considered system. In addition, the Fourier transform of $G_\rho(t-t')$ can be introduced by

$$G_\rho(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, G_\rho(t)\exp\{\mathrm{i}(\omega + \mathrm{i}\rho 0^+)t\} \qquad (2)$$

and the spectrum theorem of the Green functions tells us

$$\langle BA\rangle = \tfrac{1}{4}(1-\eta)C^{(-\eta)} + \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \frac{G_r(\omega+\mathrm{i}0^+) - G_a(\omega - \mathrm{i}0^+)}{\mathrm{e}^{\beta\omega}+\eta} \qquad (3)$$

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where $\beta = 1/k_B T$, $C^{(-\eta)} = \lim_{\omega\to 0} \omega G(\omega)$. As can easily be seen, if the Hamiltonian and the commutator of two physical quantities of a physical system are known, then we can conveniently determine the thermodynamic statistical average of the two quantities at finite temperatures in terms of (1)-(3). Following this idea, in a recent paper [5] we constructed a general approach to deriving the statistical distribution function, which is suitable for not only the usual Bose and Fermi functions but also the generalized one so long as the conditions mentioned above are satisfied. Without loss of generality of the following discussions we define the statistical distribution function as follows

$$f_\mu = \langle a_\mu^+ a_\mu \rangle \tag{4}$$

where the Hamiltonian of the system can be assumed to be presented by the operators $a_\mu^+$ and $a_\mu$. Therefore, if we know the commutator of $a_\mu^+$ and $a_\mu$ and the explicit form of the Hamiltonian expressed by $a_\mu^+$ and $a_\mu$, then the statistical distribution function $f_\mu$ can easily be derived. Below we will use this method to obtain the statistical distribution function $f_q$ for the $q$-deformed harmonic oscillator in quantum group $SU_q(2)$.

For the bosonic $q$-oscillator case, the $q$-Hamiltonian has the form [2]

$$\mathcal{H}_q = \tfrac{1}{2}\hbar\omega(a_q^+ a_q + a_q a_q^+) \tag{5}$$

where the operators $a_q^+$ and $a_q$ are constrained to obey

$$\begin{aligned} &[a_q, a_a^+]_q \equiv a_q a_q^+ - q^{1/2} a_q^+ a_q = q^{-N_q/2} \\ &[a_1^+, a_q^+]_- = [a_q, a_q]_- = 0 \end{aligned} \tag{6}$$

with $N_q$ the number operator whose eigenvalue can be supposed to be $N$, and that parameter $p$ is a real number. We should point out here that such a bosonic $q$-deformation of (6) does not affect the holomorphic property of the system. Now let us define the statistical distribution function for the $q$-deformed oscillator as

$$f_q \equiv \langle a_q^+ a_q \rangle. \tag{7}$$

To obtain $f_q$ we use $a_q^+$ and $a_q$ to construct the retarded Green function

$$G_r(t-t') = -\frac{\mathrm{i}}{\hbar}\,\theta(t-t')\langle [a_q(t), a_q^+(t')]_q \rangle \tag{8}$$

where $a_q(t) = \exp(\mathrm{i}\mathcal{H}_q t) a_q \exp(-\mathrm{i}\mathcal{H}_q t)$. From (5) and (6), we get

$$[a_q, \mathcal{H}_q]_- = \mathcal{G}(q) a_q \tag{9}$$

with

$$\mathcal{G}(q) = \tfrac{1}{2}\hbar\omega(1+q^{1/2}) \frac{\cosh\{\frac{1}{4}(2N_q+1)\ln q\}}{\cosh\{\frac{1}{4}\ln q\}} \tag{10}$$

where we have used that fact that $a_q^+ a_q = [n]$ is invariant under the duality transformation $q \leftrightarrow q^{-1}$ in the quantum group $SU_q(2)$ [6]. From (8) and (9), and by means of the equation of motion of the Green functions, we can easily obtain

$$G_r(\omega \pm \mathrm{i}0^+) = \frac{q^{-N/2}}{\omega - \mathcal{G}(q) \pm \mathrm{i}0^+}. \tag{11}$$

In the above calculation we have applied the eigenstate of the number operator $N_q$ as a complete basis to obtain the eigenvalue $N$ of $N_q$ instead of $N_q$ in $\mathscr{G}(q)$ of (11). By using (11) and (3), we can exactly derive $f_q$ for the bosonic $q$-deformed oscillator

$$f_q = \frac{q^{-N/2}}{e^{\beta\mathscr{G}(q)} - q^{1/2}}. \tag{12}$$

Evidently, when $q \to 1$, $\mathscr{G}(q) \to \hbar\omega$, then $f_q$ reproduces the usual Bose distribution function. For $q \neq 1$, $f_q$ in (12) is a general form for the bosonic $q$-oscillator. We should state that (12) is exact, and no approximations are applied to it.

Similarly, for the fermionic $q$-oscillator case [6] in a generalized sense, the Hamiltonian of the system is supposed to be

$$\mathscr{H}_q = \tfrac{1}{2}\hbar\omega(b_q^+ b_q - b_q b_q^+) \tag{13}$$

while the commutator for $b_q^+$ and $b_q$ becomes

$$\begin{aligned} \{b_q, b_q^+\}_q &= b_q b_q^+ + q^{1/2} b_q^+ b_q = q^{N/2} \\ \{b_q, b_q\} &= \{b_q^+, b_q^+\}_q = 0. \end{aligned} \tag{14}$$

We see that (13) is a fermionic oscillator-like Hamiltonian when $q \to 1$. In fact it does not possess any oscillating property in true physics, because the canonical momentum and the canonical position of the system cannot be well defined in terms of the operators $b_q$ and $b_q^+$ to satisfy the uncertainty relation. Here the introduction of (13) is only of interest in the study of the quantum group $SU_q(2)$. Analogously to the calculations of (12), we can also exactly obtain $f_q$ for the fermionic $q$-deformed oscillator

$$f_q = \frac{q^{N/2}}{e^{\beta\mathscr{F}(q)} + q^{1/2}} \tag{15}$$

where

$$\mathscr{F}(q) = \tfrac{1}{2}\hbar\omega(1 + q^{1/2})q^{N/2}. \tag{16}$$

In the limit $q \to 1$, $\mathscr{F}(q) \to \hbar\omega$, then $f_q$ decouples to the usual Fermi function. For $q \neq 1$, (15) is expected to be a general form for the fermionic $q$-oscillator.

Equations (12) and (15) are main results of this letter, from which we can see that after the $q$-deformation the corresponding distribution functions for bosons and fermions are highly variable with the parameter $q$, and are obviously different from the usual Bose and Fermi functions. We can therefore observe that the parameter $q$ in the quantum group $SU_q(2)$ plays an important role in the statistical characteristics of the $q$-oscillator. Finally, we should point out that the results of $f_q$ achieved above can be applied to an anyon gas when $q$ takes some particular forms. A detailed analysis for this point will be presented elsewhere.

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